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Local duality, chiral supersymmetry and fermion-like formulation of non-Abelian pure gauge fields

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Abstract. This work generalizes the fermion-like formulation of the Maxwell theory to the non-Abelian Yang–Mills theory without matter fields. This is a new representation of the Lie algebra valued electric and magnetic fields. The resulting equations of motion are invariant under the chiral transformation. In this formulation, duality is a kind of chirality. We may also define local duality transformations in terms of space-time dependent parameters. There is an $N = 1$ supersymmetry for the Dirac-like operator in this representation.

1 Introduction

Over the past few years, our understanding of the nonperturbative behavior of some supersymmetric field theories and also of string theory has undergone a dramatic change. This development has been motivated by the discovery of duality symmetry in supersymmetric and string theories.

Exact electromagnetic duality was first proposed in its modern form by Montonen and Olive. This duality interchanges electric charges with magnetic charges, relating strong couplings with weak ones [1,2]. There are also duality symmetries in string theories such as target space duality (T-duality) and the S-duality (the generalization of the electric–magnetic duality) [3,4]. Interesting cases also exist in the semiclassical view of electrically and magnetically charged black holes [5].

Among these dualities the most familiar one is the electric–magnetic (EM) duality of the Maxwell equations. The simplest case is the free Maxwell equations in which the EM duality is a symmetry of these equations under rotations of the electric and magnetic fields. It is a symmetry between the equations of motion and the Bianchi identities. The fermion-like formulation for the electromagnetic theory has already been studied extensively [8–10]. In this paper, efforts will be made to generalize the fermion-like formulation of Majorana [6,7] for the Maxwell theory to the non-Abelian Yang–Mills theory. In this formulation of non-Abelian pure gauge fields, duality is a kind of chirality and the equations of motion are invariant under the duality transformations. It is also possible to define local duality trasformations as a result of the new formulation. The Dirac-like operator in this representation has $N = 1$ supersymmetry.

This paper is organized as follows: in Sect. 2, a fermion-like formulation for the non-Abelian gauge fields is introduces. In Sect. 3, the relation between chirality and duality is explained. In Sect. 4, it is shown that the square of the Dirac-like operator has a chiral supersymmetry.

2 Fermion-like formulation of non-Abelian gauge fields

Feynman once said that "every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics" [13]. In this part the already established fermion-like formulation of Maxwell theory is generalized to the non-Abelian pure gauge fields. We introduce Lie algebra valued gauge fields by $A_\mu = A^a_\mu T^a$, and their field strengths as follows:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{1}$$

The antihermitian matrices T^a satisfy the algebra

$$[T^a, T^b] = f^{abc} T^c, \tag{2}$$

and they are normalized by $\mathrm{tr} T^a T^b = -1/(2)\delta^{ab}$. In the absence of matter fields, in the case of a pure Yang–Mills theory we have

$$D_\mu F^{\mu\nu} = 0, \tag{3}$$

$$D_\mu \widetilde{F}^{\mu\nu} = 0, \tag{4}$$

where

$$D_\mu = \partial_\mu + [A_\mu, \ \ ], \tag{5}$$

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and

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \quad (\epsilon^{0123} = 1). \tag{6}$$

We define Lie algebra valued electric and magnetic fields by

$$E^i = F^{i0}, \tag{7}$$

$$B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk}. \tag{8}$$

Equation (3) for $\nu = i$ can be rewritten as

$$D_0 F^{0i} + D_j F^{ji} = 0, \tag{9}$$

so that the Lie algebra valued electric (7) and the magnetic (8) fields may be replaced in (9) to obtain (10),

$$\begin{aligned} D_0 E^i &= \epsilon^{jik} D_j B_k \\ &= \mathrm{i}(S^j)^{ik} D_j B_k \\ &= \mathrm{i}(\boldsymbol{S}.\boldsymbol{D})^{ik} B_k, \end{aligned} \tag{10}$$

where $(S^i)^{jk} = -\mathrm{i}\epsilon^{ijk}$ provides us with 3×3 matrices, and where ϵ^{ijk} is the Levi-Civita totally antisymmetric tensor, normalized as $\epsilon^{123} = 1$. The S-matrices are as follows:

$$S^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad S^2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix},$$
$$S^3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

Equation (10) can be rewritten as

$$D_0 \widehat{E} = -(\boldsymbol{S}.\boldsymbol{D})\widehat{B}. \tag{12}$$

Using (4), it is straightforward to see that

$$D_0 \widehat{B} = -(\boldsymbol{S}.\boldsymbol{D})\widehat{E}, \tag{13}$$

where

$$\widehat{E} = \begin{pmatrix} E^1 \\ E^2 \\ E^3 \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} \mathrm{i}B^1 \\ \mathrm{i}B^2 \\ \mathrm{i}B^3 \end{pmatrix}. \tag{14}$$

In order to write (12) and (13) in a fermion-like formulation, the five 6×6 matrices defined in [10] are used,

$$\Gamma^i = \begin{pmatrix} 0 & S^i \\ -S^i & 0 \end{pmatrix},$$
$$\Gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{15}$$

where I is a 3×3 identity matrix, Γ^i and Γ^0 are similar to the Dirac gamma matrices, noting that they do not obey the usual gamma anticommutation relations. However, Γ^5 anticommutes with the other gamma matrices. We have

$$\{\Gamma^5, \Gamma^\mu\} = 0 \quad (\mu = 0, .., 3). \tag{16}$$

Now a wave function for the gauge fields may be defined by

$$\psi = \begin{pmatrix} \widehat{E} \\ \widehat{B} \end{pmatrix}. \tag{17}$$

Therefore, (12) and (13) may be written in a compact form,

$$\Gamma^\mu D_\mu \psi = 0. \tag{18}$$

This is similar to the equation of motion for a fermion that has interaction with a gauge field. Actually (3) and (4) can be represented by two equations, one a curl equation and the other a divergence equation [11]. Equation (18) is the curl part of (3) and (4). It is also possible to find another equation which implies the divergence parts of (3) and (4). Let us first consider the simple case where (18) for the Abelian gauge fields is reduced to

$$\Gamma^\mu \partial_\mu \psi = 0. \tag{19}$$

This is equivalent to the Maxwell curl equations. Consider the following equation:

$$\Gamma^\nu \partial_\nu \Gamma^\mu \partial_\mu \psi = 0, \tag{20}$$

in which the Γ^μ-matrices do not obey the usual gamma anticomutation relations and one cannot get to the massless Klein–Gordon wave equation. By adding the two Maxwell divergence equations to (20), however, the Klein–Gordon wave equation for ψ may be obtained which is different from (19). We have

$$\partial^\mu \partial_\mu \psi = 0. \tag{21}$$

Using the same method, a non-linear wave equation can be obtained for the Lie algebra valued electric and magnetic gauge fields, or for ψ,

$$(D^\mu D_\mu + \Gamma^\mu \Gamma^\nu [D_\mu, D_\nu])\psi = 0. \tag{22}$$

Equation (18) together with (22) form a complete set of equations for describing a classical (electromagnetic) gluonic gauge field. Equation (22) reduces to the linear wave equation (21) for the Abelian gauge fields. The interesting point about (18) and (19) is that their conserved current can be defined by

$$j^\mu = \bar{\psi}\Gamma^\mu\psi. \tag{23}$$

j^μ is exactly equal to the canonical energy momentum tensor $\Theta^{0\mu}$ which is a conserved quantity. The other conserved current for the massless Dirac-like equation (19) is $j^5_\mu = \bar{\psi}\Gamma_\mu\Gamma^5\psi$ which is identically equal to zero in this case.

3 Duality transformations

Equations (18) and (22) are invariant under the following chiral transformation:

$$\psi \to \psi' = \mathrm{e}^{\mathrm{i}\theta\Gamma_5}\psi, \tag{24}$$

where θ is a constant. One can expand the exponential phase factor and write it as a 6×6 matrix:

$$\begin{pmatrix} \widehat{E}' \\ \widehat{B}' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \widehat{E} \\ \widehat{B} \end{pmatrix}. \tag{25}$$

Therefore, the equations of motion are invariant under rotations of the Lie algebra valued electric and magnetic fields. For example, at $\theta = -\pi/2$ electric and magnetic fields interchange to produce the simplest kind of duality for non-Abelian pure gauge fields. Considering the following local transformations, a local duality transformation can be defined as follows:

$$\psi \to \psi' = g^{-1}(x)\psi, \tag{26}$$

$$D_\mu \to g D_\mu g^{-1}, \tag{27}$$

where $g^{-1}(x) = \mathrm{e}^{\mathrm{i}\theta(x)\Gamma_5}$. The result from transformation (26) is a mixing of the Lie algebra valued electric and magnetic fields through a space-time dependent angle. The transformation may be called a gauge-like transformation, but it should be noted that it is not a gauge transformation for the fields. A Lagrangian formulation may be used to obtain the basic equation (18),

$$L(\psi, \bar{\psi}, D_\mu\psi, D_\mu\bar{\psi}) = \mathrm{tr}\bar{\psi}\Gamma^\mu D_\mu\psi. \tag{28}$$

The Lagrangian is invariant under the chiral (duality) transformations (24) and the equation of motion for ψ can be obtained from the Euler–Lagrange equation. We have

$$\frac{\partial L}{\partial\bar{\psi}} - D_\mu\left(\frac{\partial L}{\partial D_\mu\bar{\psi}}\right) = 0 \tag{29}$$

$$\Rightarrow \frac{\partial L}{\partial\bar{\psi}} = 0 \tag{30}$$

$$\Rightarrow \Gamma^\mu D_\mu\psi = 0. \tag{31}$$

4 $N = 1$ chiral supersymmetry

The interesting point about the curl part (18) of the Yang–Mills field equations is that the Dirac-like operator in (18) has an $N = 1$ supersymmetry. Consider the following definitions:

$$H = (\Gamma^\mu D_\mu)^2, \tag{32}$$

$$Q_\pm = \frac{1}{2}(1 \pm \Gamma^5)\Gamma^\mu D_\mu. \tag{33}$$

It can easily be seen that operators $Q_\pm$ and H satisfy the usual $N = 1$ SUSY algebra,

$$H = \{Q_+, Q_-\}, \tag{34}$$

$$\{H, Q_\pm\} = 0. \tag{35}$$

This is known as $N = 1$ chiral supersymmetry [12]; it has not been noticed for the curl part of the equations of motion for the Yang–Mills gauge fields.

5 Conclusion

In this work, the Majorana formulation of Maxwell theory was generalized to the Yang–Mills theory. The invariance of the field equations under a chiral transformation was used to make global and local duality transformations. In this representation, the duality transformation was a kind of chirality with a simple form. Also a new non-linear wave equation, (22), was developed for the Lie algebra valued electric and magnetic fields. It will be interesting to investigate the conserved currents associated with this non-linear wave equation.

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